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Comparison of the Lift to Drag Ratio for Wave Riders with Different Shape and Angle of Attack via Perturbation Method

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Abstract: An Analyze is performed to study the hypersonic flow over wave riders as conical bodies with different cross sections and longitudinal curvatures at different angles of attack. The calculation of lift to drag ratio of these cross sections and comparison of them with cones with no longitudinal curvature and different longitudinal curvatures is major subject in this research. Perturbation expansion is considered for flow variables and is used in terms of ε as a perturbation parameter and α as a attack angle. In hypersonic flow, the boundary layer is very thin and viscous effects are negligible and flow is assumed to be adiabatic. Governing equations are conservation of mass, momentum, energy (in two forms of entropy and enthalpy), and equation of state. The zeroth-order approximation of hypersonic conical flow obtain by nonlinear asymptotic theory is chosen as the basic-cone solution for expansions. In this analysis, the complicated governing equation of flow field can be simplified by an appropriate approximation scheme, and the first-order approximations of properties are derived. With small angle assumption final equation is reduced to the simple form of radial velocity. With solving this equation all the flow variables for the shock layer flow field can be evaluated. As a major parameter in design of aircrafts and space vehicles the lift to drag ratio is calculated. Results show changing cross section from circle to squirrel (rounded square) increases the lift to drag ratio. Also presence of longitudinal curvature effect increases this ratio. These results indicate that studied cones can be used effectively to efficiently integrate propulsion and aerodynamic requirements for a variety of hypersonic vehicles.

Keywords: Hypersonic flow, perturbation method, longitudinal curvature, wave rider.

مقایسه نسبت نیروی برا به کشش برای موج‌ران با شکل و زاویه حمله متفاوت با استفاده از روش پرتوربیشن

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چکیده: در این مقاله تحلیلی بر روی پرتابه فراصوتی در حال حرکت داخل یک لوله شوک انجام شده‌است. این پرتابه دارای انحناء در سطح‌جانبی بوده و با سه سطح مقطع؛ دایره، بیضی و مثلث بررسی می‌شود. این سطوح مقطع توسط اغتشاش مخروطی شکل به‌عنوان مخروط پایه ایجاد می‌شوند. در این مطالعه برای تعیین پارامتر شوک و زاویه شوک از روش پرتوربیشن استفاده شده‌است. در این پوهش با کاربرد یک روش تقریبی تحلیلی معادلات پیچیده حاکم بر میدان جریان ساده می‌شوند و سعی براین است که با استفاده ازبسط پرتوربیشن، حل تحلیلی معتبری را به صورت یکنواخت در لایه شوک به‌دست آورد. پرتابه‌های اقتباس یافته از میدان جریان اغتشاشی زاویه شوک مناسب را تولید کرده و مزیت‌هایی از نظر طراحی برای لوله شوک فراهم می‌آورند. بررسی دقیق نتایج، اثر سطح‌مقطع را بر زاویه شوک و پارامتر شوک نشان می‌دهند. همچنین مشاهده می‌شود با افزایش نیم زاویه مخروط، زاویه شوک افزایش می‌یابد. این مطالعه، همچنین برای بررسی و درک میدان جریان در شتاب دهنده‌ها و شلیک گلوله روش موثری را ارائه می‌نماید که ممکن است طراحی مناسبی را برای پرتاب یک پرتابه فراصوتی ایجاد کند.

1. Introduction

One way for studying the aerodynamics of these configurations is by means of conical bodies. These configurations prepare suitable flow fields and aerodynamic properties. The flow past conical bodies has been studied for many different cases. Perturbation method is widely applied to study of flow on conical bodies. Stone [1] applied the power series expansion for a small attack angle and obtained the first- and second-order perturbation. The analysis is similar to that of Doty and Rasmussen [2] for obtaining solutions for flow past circular cones at small angle of attack. Starting from the zero-order approximation of a hypersonic conical flow obtained by nonlinear asymptotic theory [3] is adopted as the basic-cone solution for the outer expansion.

The purpose of the present work is to compare the lift to drag ratio for different cross sections analytically. The results will be useful in increasing the lift to drag ratio for aircrafts, missiles and space vehicles by changing cross section and longitudinal curvature.

2. Formulation and governing equations

Normal spherical polar coordinates r, θ and ϕ are used in this study as shown in Fig. (1), θ is the polar angle and ϕ is the azimuth angle. In this perturbed flow, the next expression represents the velocity vector:

$$\vec{V} = u\,\hat{e}_r + v\,\hat{e}_\theta + w\,\hat{e}_\Phi \tag{1}$$

The basic cone body is perturbed by the relation of:

$$\theta_c = \delta[1-\varepsilon_m(\frac{r}{l})^m \cos n\phi] \tag{2}$$

Where δ is the semi-vertex angle of basic cone, l is cone length; ε is a small perturbation parameter. Longitudinal curvature is denoted by $(r/l)^m$ where m is integer value and with increasing of it longitudinal curvature increases as shown in Fig.(2). When m = 0, it represents a cone has no the longitudinal curvature. Cross section effects are represented by cos(nϕ) that n is 0, 2 and 4 for circular, elliptical and squirrel cross section, respectively that are seen in Fig.(3). The shape of the corresponding shock wave is expressed in a similar way as:

$$\theta_s = \delta[1-\varepsilon_m G_m(\frac{r}{l})^m \cos n\varphi + o(\varepsilon^2)] \tag{3}$$

Where G_m shows the shock-perturbation factor, and σ =β/δ is the ratio of shock angle to body angle for the basic cone. Here we assume the flow, outside the viscous boundary layer, is inviscid, adiabatic, and steady. Thus the governing equations expressing conservation of mass, momentum, and energy (in both entropy and enthalpy forms) can be written as:

$$\mathrm{div}(\rho\vec{V}) = 0 \tag{4}$$

$$\rho[\nabla(\frac{V^2}{2}) - \vec{V}\times \mathrm{curl}\vec{V}] = -\nabla p \tag{5}$$

$$\vec{V}.\nabla s = 0 \tag{6}$$

$$\frac{\gamma}{\gamma-1}\frac{p}{\rho} + \frac{u^2+v^2+w^2}{2} = \mathrm{cons.} \tag{7}$$

$$\frac{s-s_r}{C_v} = \mathrm{Ln}(\frac{p}{p_r}) - \gamma \mathrm{Ln}(\frac{\rho}{\rho_r}) \tag{8}$$

Where, S_r, p_r, and ρ_r are suitable reference quantities, C_v and γ are specific heat capacity at constant volume and heat capacity ratio, respectively. We changed the shape of basic cone by perturbation variables, thus the conventional spherical coordinate system is not valid in the perturbed cone and the perturbed shock. It is necessary to establish a new coordinate variable $\theta_o = \theta_o(r, \theta, \phi)$ to exchange the polar angle θ between the cone surface and shock layer by definitions. θ_o is such that:

$$\frac{\theta_o-\delta}{\beta-\delta} = \frac{\theta-\theta_c(r,\phi)}{\theta_s-\theta_c(r,\phi)} \tag{9}$$

When $\theta = \theta_c(r,\phi)$, is located on the actual cone surface, it is seen that the new variable will be $\theta_o = \delta$, when $\theta = \theta_s(r, \phi)$ it will be $\theta_o = \beta$. We know that longitudinal curvature is proportional to the ε_m assumed to be small. Thus, we suppose the first order expansion for a perturbed flow can be expressed as [4].

$$u = u_o(\theta_o) + \varepsilon u_m(\theta_o)(r/l)^m \cos(n\phi) + o(\varepsilon^2) \quad (10)$$

$$v = v_o(\theta_o) + \varepsilon v_m(\theta_o)(r/l)^m \cos(n\phi) + o(\varepsilon^2) \quad (11)$$

$$w = \varepsilon w_m(\theta_o)(r/l)^m \sin(n\phi) + o(\varepsilon^2) \quad (12)$$

$$p = p_o(\theta_0) + \varepsilon p_m(\theta_o)(r/l)^m \cos(n\phi) + o(\varepsilon^2) \quad (13)$$

$$\rho = \rho_o(\theta_o) + \varepsilon \rho_m(\theta_o)(r/l)^m \cos(n\phi) + o(\varepsilon^2)$$
(14)

$$s = s_o(\theta_o) + \varepsilon s_m(\theta_o)(r/l)^m \cos(n\phi) + o(\varepsilon^2) \quad (15)$$

In these expansions zero order functions $u_o(\theta_o)$, $v_o(\theta_o)$, $w_o(\theta_o)$, $p_o(\theta_o)$, $\rho_o(\theta_o)$, and $s_o(\theta_o)$, are the solutions of basic cone problem. In this study, the zero order solution from [3] is adopted as the basic solution and they for the perturbed flow field are:

$$u_o(\theta_o) \cong V_\infty[1 - \frac{\delta^2}{2}(\frac{\theta_o^2}{\delta^2} + \ln\frac{\beta^2}{\theta_o^2})] \quad (16)$$

$$v_o(\theta_o) \cong -V_\infty \theta_o (1 - \frac{\delta^2}{\theta_o^2}) \quad (17)$$

$$\frac{a_o^2(\theta_o)}{a_\infty^2} \equiv \frac{T_o(\theta_o)}{T_\infty} \cong 1 + \frac{\gamma - 1}{2} k_\delta^2 [2 - \frac{\delta^2}{\theta_0^2} + \ln\frac{\beta^2}{\theta_o^2}] \quad (18)$$

$$\frac{\rho_o(\theta_o)}{\rho_\infty} = \frac{\rho_o(\beta)}{\rho_\infty}[1 + \frac{k_\delta^2}{2(T_o(\beta)/T_\infty)}(\frac{\delta^2}{\beta^2} - \frac{\delta^2}{\theta_o^2} + \ln\frac{\beta^2}{\theta_o^2})] \quad (19)$$

$$\frac{p_o}{p_\infty} \cong 1 + \frac{\gamma}{2} k_\delta^2 [1 + \frac{\rho_o(\beta)}{\rho_\infty}(1 - \frac{\delta^2}{\theta_o^2} + \ln\frac{\beta^2}{\theta_o^2})] \quad (20)$$

Where:

$$\xi_o \equiv \frac{\rho_\infty}{\rho_o(\beta)} \cong 1 - \frac{1}{\sigma^2} \quad (21)$$

And $\sigma \equiv \frac{\beta}{\delta} \cong [\frac{\gamma+1}{2} + \frac{1}{k_\delta^2}]^{1/2}, \quad k_\delta = M_\infty \delta$ (22)

3. The perturbation of boundary condition

The body surface $\theta_o = \delta$, boundary condition can be determined by the tangency of flow across the body surface as:

$$\vec{V}_c . \hat{n}_c = 0 \quad (23)$$

Where, $\hat{n}_c$ is the unit vector that is outward normal from the cone surface and is defined by:

$$\hat{n}_c = \varepsilon r \delta f'(r/l)\cos(n\phi)\hat{e}_r + \hat{e}_\theta - \frac{1}{\sin\delta}[\varepsilon\delta f(r/l) n \sin(n\phi)]\hat{e}_\phi \quad (24)$$

When we substitute the Eqs. (10) and (11) into the (23), zero and first order is obtainedas:

$$v_o(\delta) = 0 \text{ and } v_m(\delta) = -mV_\infty \delta \quad (25)$$

With the equations of mass and tangential velocity conservation, velocity components can be calculated at the shock $\theta = \beta$:

$$\begin{cases} V_\infty \times \hat{n}_s = V_s \times \hat{n}_s \\ \rho_\infty V_\infty . \hat{n}_s = \rho V_s . \hat{n}_s \end{cases} \quad (26)$$

Substituting the expressions (11-15) into above equations the zero and first order results for boundary conditions can be obtained as follows:

$$\frac{u_o(\beta)}{V_\infty} = \cos\beta \quad (27)$$

$$\frac{v_o(\beta)}{V_\infty} = -\xi_o \sin\beta \quad (28)$$

$$u_m(\beta) = \delta^2 V_\infty \sigma G_m (1 + \frac{m}{\sigma^2}) \quad (29)$$

$$\frac{v_m(\beta)}{V_\infty} = \delta G_m \cos\beta[(1+m)\{\frac{2(\gamma-1)}{\gamma+1} - \xi_o\} - m] \quad (30)$$

$$w_m(\beta) = n(\xi_o - 1)\delta G_m V_\infty \quad (31)$$

$$\frac{s_o(\beta) - s_\infty}{C_v} = \mathrm{Ln}\frac{p_o(\beta)}{p_\infty} + \gamma \mathrm{Ln}\xi_o \quad (32)$$

$$s_m(\beta) = (\frac{p_m(\beta)}{p_o} + \gamma \frac{\xi_m}{\xi_o}) \quad (33)$$

Where:

$$\xi_m = 2\delta \cot\beta(1 - G_m)(\xi_o - \frac{\gamma - 1}{\gamma + 1}) \quad (34)$$

3.1 Energy Equation

Substitution of expansions into equation (8) and separation of various orders of perturbation lead to:

$$s_o = 0 \quad (35)$$

$$\frac{ds_m}{d\theta_o} + m\frac{u_o}{v_o}s_m = 0 \quad (36)$$

3.2 Pressure and Density

When expansions are substitute into the Bernoulli equation (7) and state equation (8), the first order perturbation will be:

$$\frac{a_o^2}{\gamma - 1}(p_m - \rho_m) + u_o u_m + v_o v_m = 0 \quad (37)$$

$$p_m - \gamma\rho_m = s_m \quad (38)$$

The first order perturbation for pressure p_m and ρ_m are in the form:

$$p_m(\theta_o) = -\frac{\gamma(u_o u_m + v_o v_m)}{a_o^2} - \frac{s_m}{(\gamma - 1)} \quad (39)$$

$$\rho_m(\theta_o) = -\frac{(u_o u_m + v_o v_m)}{a_o^2} - \frac{s_m}{(\gamma - 1)} \quad (40)$$

The r, θ and ϕ momentum equations are expanded to the zero and first order using expansions (10) to (15):

First order perturbation for r-momentum equation (4) can be written as:

$$u'_m + mu_m\frac{u_o}{v_o} - v_m + \frac{p_m m}{\rho_o v_o} - v_0\frac{d\theta_m}{d\theta_o} - u_o m\theta_m - \frac{m\theta_m p'_o}{\rho_o v_o} = 0 \quad (41)$$

Inserting Eq. (39) into (41), generates v_m as a function of u_m and θ_m,

$$v_m = \frac{1}{1+m}[\frac{du_m}{d\theta_o} - \frac{m}{\gamma(\gamma-1)}\frac{a_o^2}{v_o}s_m - v_o\frac{d\theta_m}{d\theta_o} + m\frac{dv_o}{d\theta_o}\theta_m] \quad (42)$$

In this study we introduce new variables for simplification and to eliminate $d\theta_m/d\theta_o$. It is accomplished by inserting the following velocity transformations [4]:

$$u_m^*(\theta_o) = u_m - \theta_m(\theta_o)v_o(\theta_o) \quad (43)$$

$$v_m^*(\theta_o) = v_m - \theta_m(\theta_o)\frac{dv_o(\theta_o)}{d\theta_o} \quad (44)$$

Inserting new variables into the Eq. (42), yields,

$$v_m^* = \frac{1}{m+1}[\frac{du_m^*}{d\theta_o} - \frac{m}{\gamma(\gamma-1)}\frac{a_o^2}{v_o}s_m] \quad (45)$$

3.3 Azimuthal Velocity Component

The first order perturbation of azimuthal velocity is obtained via ϕ momentum equation. By substituting Eq. (39) in the first order perturbation terms it yields:

$$n\frac{u_o}{v_o}u_m + (1+m)\frac{u_o}{v_o}\sin\theta_o + w_m\cot\theta_o + w'_m\sin\theta_o + \frac{n}{\gamma(\gamma-1)}\frac{a_o^2}{v_o}s_m - n\theta_m v'_o - n\theta_m u_o = 0 \quad (46)$$

Taking the new transformation (43) and (44) and substituting Eq. (45) into expression (46) generates the first order azimuthal perturbation as a function of u^*_m:

$$(w_m \sin\theta_o + \frac{n}{m+1} u_1^*)' + \frac{u_o}{v_o}((1+m) w_m \sin\theta_o + nu_m^*) + \frac{n}{\gamma(\gamma-1)} \frac{a_o^2}{v_o} s_m (\frac{1}{1+m}) = 0 \tag{47}$$

Integrating the above equation gives the azimuthal velocity perturbation as a function of u^*_m.

3.4 Continuity Equation

The zero order perturbation for r and θ momentum equations yields respectively,

$$v_o^2 - u_o' v_o = 0 \tag{48}$$

$$\rho_o' - \rho_o(-2\frac{u_o}{v_o} - \cot\theta_o - \frac{v_o'}{v_o}) = 0 \tag{49}$$

Adding zero order of r and θ momentum into continuity equation (4), the zero order expansion of this equation can be written as:

$$(1 - \frac{v_o^2}{a_o^2}) v_o' + vo \cot\theta_o + (2 - \frac{v_o^2}{a_o^2}) u_o = 0 \tag{50}$$

We must find the expression of azimuthal velocity perturbation in terms of u^*_m. Adding the azimuthal velocity perturbation from integrating the expression (47) and substituting the zero order of r and θ momentum and expression (45) and Eqs. (39-40) into continuity equation (4), the zero order expansion of this equation can be written as:

$$[(1-B)\cot\theta_o + (1+m)\frac{u_o v_o}{a_o^2}] v_m + [(2+m-c) - m\frac{u_o^2}{a_o^2}] u_m + (1-A)\frac{dv_m}{d\theta_o} - \frac{u_o v_o}{a_o^2}\frac{du_m}{d\theta_o} + \frac{n}{\sin\theta_o} w_m = \frac{v_o}{\sin^2\theta_o}\theta_m + \frac{1}{\rho_m}\frac{d(\rho_o v_o)}{d\theta_o}\frac{d\theta_m}{d\theta_o} \tag{51}$$

With:

$$A(\theta_o) = \frac{v_o^2}{a_o^2} \tag{52}$$

$$B(\theta_o) = \tan\theta_o [\frac{v_o}{a_o^2}(\frac{dv_o}{d\theta_o} + u_o) - \frac{d}{d\theta_o}(Ln\rho_o) - \frac{v_o^2}{a_o^2}\frac{d}{d\theta_o}(Ln a_o^2)] \tag{53}$$

$$v_m = \frac{1}{1+m}[\frac{du_m}{d\theta_o} - \frac{m}{\gamma(\gamma-1)}\frac{a_o^2}{v_o} s_m - v_o \frac{d\theta_m}{d\theta_o} + m\frac{dv_o}{d\theta_o}\theta_m] \tag{54}$$

By incorporating changed variables into Eq. (51) and eliminating the term $(md(\rho_o v_o)/\rho_o d\theta_o)$, conservation of mass equation can be written:

$$(1-A)\frac{dv_m^*}{d\theta_o} + [(1-B)\cot\theta_o + (1+m)\frac{u_o v_o}{a_o^2}] v_m^* - \frac{u_o v_o}{a_o^2}\frac{du_m^*}{d\theta_o} + [(2+m+C) - m\frac{u_o^2}{a_o^2}] u_m^* + \frac{4}{\sin\theta_o} w_m = 0$$

(55)

4. Approximation shame and solution

The analysis can be simplified even more by the introducing of another variable in the form of:

$$u_m^{**} = u_m^* - \int_\delta^{\theta_o} \frac{m a_o^2}{\gamma(\gamma-1) v_o} s_m d\theta_o \tag{56}$$

By inserting this new variable into Eq. (45), the following expression is obtained:

$$v_1^* = \frac{1}{1+m}\frac{du_1^{**}}{d\theta_o} \tag{57}$$

The variation in the integral term in Eq. (56) can be treated as constant, because the change in $a_o^2(\theta_o)$, from the minimum value at the shock to its largest value at the body is always small, substituting Eq. (36) into Eq. (56), the integral can be evaluated, and Eq. (56) can be written in the form of:

$$u_m^{**} = u_m^* + \frac{a_o^2}{\gamma(\gamma-1) u_o} s_m \tag{58}$$

Applying this new variable into ϕ momentum, Eq. (47), the azimuthal component of velocity can be expressed as follows:

$$\frac{w_m}{V_\infty \delta} = -\frac{n}{1+m}\frac{\delta}{\sin\theta_o}\frac{u_m^{**}}{V_\infty \delta^2} - \frac{n\sigma G_m(\xi_o - 1)^2}{\sin\theta_o/\delta}\left(\frac{1}{I}\right)^{1+m} \quad (59)$$

Substituting Eqs. (58) and (59) into continuity Eq. (55), it takes the following form:

$$\begin{aligned}&(1-A)\frac{du_m^{**}}{d\theta_o} + [\cot\theta_o(1-B) - \frac{2mu_o v_o}{a_o^2}]\frac{du_m^{**}}{d\theta_o} + (1+m)\\ &[(2-C+m) - m\frac{u_o^2}{a_o^2} - \frac{16}{1+m}\frac{1}{\theta_o^2}]u_m^{**} = \frac{1+m}{\gamma(\gamma-1)}\\ &(2-C+m)\frac{a_o^2}{u_o}s_m + V_\infty\frac{16(1+m)}{\theta_o^2/\delta^2}G_m\sigma(\xi_o-1)^2\left(\frac{1}{I}\right)^{1+m}\end{aligned} \quad (60)$$

Which is a single linear second-order ODE for u_m^{**}, for small values of angle, it reduced to[4]:

$$\frac{du_m^{**}}{d\theta_o} + \frac{1}{\theta_o}\frac{du_m^{**}}{d\theta_o} - (C_m^2 + \frac{n^2}{\theta_o^2})u_m^{**} = H_m(\theta_o) \quad (61)$$

where,

$$H_m(\theta_o) = \frac{n^2(1+m)}{\theta_o^2/\delta^2}V_\infty\sigma G_m(\xi_o-1)^2\left(\frac{1}{I}\right)^{1+m} \quad (62)$$

$$C_m = \lambda_m/\delta \quad (63)$$

$$\lambda_m = [\frac{m(1+m)}{a_o^2(\beta)/a_\infty^2}] \quad (64)$$

The homogeneous solution for Eq. (61) is:

$$\left(\frac{u_m^{**}(\zeta)}{V_\infty\delta^2}\right)_H = G_m[A_m I_n(\lambda_m\zeta) + B_m K_n(\lambda_m\zeta)] \quad (65)$$

A particular solution must be added to Eq.(65) to get the complete the solution:

$$\begin{aligned}\left(\frac{u_m^{**}(\zeta)}{V_\infty\delta^2}\right)_P &= I_n(\lambda_m\zeta)\int_1^\zeta \zeta K_n(\lambda_m\zeta)\frac{H_m(\zeta)}{V_\infty}d\zeta - \\ &K_n(\lambda_m\zeta)\int_1^\zeta \zeta I_n(\lambda_m\zeta)\frac{H_m(\zeta)}{V_\infty}d\zeta\end{aligned} \quad (66)$$

The $X_m(\zeta)$ and $Y_m(\zeta)$ are found as:

$$X_m(\zeta) = n^2(1+m)\sigma(\xi_o-1)^2\int_1^\zeta \frac{K_n(\lambda_m\zeta)}{\zeta}\left(\frac{1}{I}\right)^{1+m} \quad (67)$$

$$Y_m(\zeta) = n^2(1+m)\sigma(\xi_o-1)^2\int_1^\zeta \frac{K_n(\lambda_m\zeta)}{\zeta}\left(\frac{1}{I}\right)^{1+m} \quad (68)$$

The general solution for Eq. (61) is:

$$\begin{aligned}\frac{u_m^{**}(\zeta)}{V_\infty\delta^2} &= I_n(\lambda_m\zeta)G_m[A_m + X_m(\zeta)] + \\ &K_n(\lambda_m\zeta)G_m[B_m - Y_m(\zeta)]\end{aligned} \quad (69)$$

Substituting the solution (69) into Eq. (57), v_m^* is obtained as:

$$\begin{aligned}\frac{v_m^*(\zeta)}{V_\infty\delta} &= \frac{\lambda_m}{1+m}G_m\{I_4'(\lambda_m\zeta)[A_m + X_m(\zeta)] + \\ &K_4'(\lambda_m\zeta)[B_m - Y_m(\zeta)]\}\end{aligned} \quad (70)$$

where, $I'_4(\lambda_m\zeta)$ and $K'_4(\lambda_m\zeta)$ are derivatives of $I_4(\lambda_m\zeta)$ and $K_4(\lambda_m\zeta)$.

The boundary conditions can be written in terms of u_m^{**} and v_m^{**} as:

$$\frac{u_m^{**}(\zeta=\sigma)}{V_\infty\delta^2} = (1+m)\sigma\xi_o G_m \quad (71)$$

$$\begin{aligned}\frac{v_m^*(\zeta=\sigma)}{V_\infty\delta} &= G_m\{(1+m)[\frac{2(\gamma-1)}{\gamma+1} - \xi_o] - m\} + \\ &\frac{G_m}{V_\infty}\frac{dv_o(\zeta=\sigma)}{d\theta_o}\end{aligned} \quad (72)$$

$$\frac{v_m^*(\zeta=1)}{V_\infty\delta} = -(m+2) \quad (73)$$

The above shock boundary conditions are substitute into Eqs. (69) and (70), respectively to solve A_m and B_m:

$$A_m = N_m - X_m(\zeta=\sigma) \quad (74)$$

$$B_m = Q_m + Y_m(\zeta=\sigma) \quad (75)$$

where:

$$N_m = \frac{-(1+m)\sigma}{G_m}\left[\frac{\lambda_m}{1+m}K'_n(\lambda_m\sigma)\frac{u_m^{**}(\zeta=\sigma)}{V_\infty\delta^2} - K_n(\lambda_m\sigma)\frac{v_m^{*}(\zeta=\sigma)}{V_\infty\delta}\right] \quad (76)$$

$$Q_m = \frac{1}{K_n(\lambda_m\sigma)G_m}\frac{u_m^{**}(\zeta=\sigma)}{V_\infty\delta^2} - \frac{I_n(\lambda_m\sigma)}{K_n(\lambda_m\sigma)}N_m \quad (77)$$

Substituting the surface boundary condition (73) into Eq. (70), the value of shock perturbation parameter can be obtained as:

$$G_m = \frac{1+m}{\lambda_m(I'_n(\lambda_m)A_m + K'_n(\lambda_m)B_m)}\frac{v_m^{*}(\zeta=1)}{V_\infty\delta} \quad (78)$$

To achieve a complete solution for flow over a conical body at small angle of attack another perturbation expansion should be written for flow variables in which α (angle of attack) is the perturbation factor,

$$u(\theta,\phi,\alpha) = u_o(\theta) + \alpha u_2(\theta)\cos\phi + o(\alpha^2) \quad (79)$$

$$v(\theta,\phi,\alpha) = v_o(\theta) + \alpha v_2(\theta)\cos\phi + o(\alpha^2) \quad (80)$$

$$w(\theta,\phi,\alpha) = \alpha w_2(\theta)\sin\phi + o(\alpha^2) \quad (81)$$

$$p(\theta,\phi,\alpha) = p_o(\theta) + \alpha p_2(\theta)\cos\phi + o(\alpha^2) \quad (82)$$

$$\rho(\theta,\phi,\alpha) = \rho_o(\theta) + \alpha \rho_2(\theta)\cos\phi + o(\alpha^2) \quad (83)$$

$$s(\theta,\phi,\alpha) = s_o(\theta) + \alpha s_2(\theta)\cos\phi + o(\alpha^2) \quad (84)$$

The parameters with index 2 are related with flow over circular cone with angle of attack. In this case the equation of the body is:

$$\theta_c = \delta + \alpha\cos\phi + o(\alpha^2) \quad (85)$$

Substitute the perturbation expansions with respect to α in the governing equations, and separate zero and first order terms in α, two systems of equations are obtained. The boundary conditions for first order can be obtained as:

$$u_2(\beta) = \delta\sin\beta\left(1 - G_2(1-\xi_o)\right) \quad (86)$$

$$u'_2(\beta) = -\delta G_2 v'_o(\beta) + \delta\xi_o\cos\beta\left(1-g_2\right) - \xi_2\sin\beta \quad (87)$$

$$v_2(\delta) = 0 \quad (88)$$

Where G_2 represents shock perturbation parameter for flow passes with angle of attack on circular cone. The second system of equations leads to the following ODE with respect to u_2.

$$u''_2 + u'_2\cot\theta + u_2\left(2 - \frac{1}{\sin^2\theta}\right) = -\frac{F_2}{\gamma}\frac{H_o(\theta)}{\sin^2\theta} \quad (89)$$

For small angles the above ODE has a solution of:

$$\frac{u_1(z)}{\delta^2} = G_{11}Z - G_{12}\frac{1}{Z} + G_{13}R \quad (90)$$

$$\frac{v_1(Z)}{\delta} = G_{11} + G_{12}\frac{1}{Z^2} + G_{13}\frac{dR}{dZ} \quad (91)$$

where:

$$G_{21} = \frac{1}{2\sigma^2} + \frac{\gamma-1}{\gamma+1} + g_2\left(\frac{2}{\gamma+1} - \frac{1}{2\sigma^2}\right) \quad (92)$$

$$G_{22} = \left(\frac{1}{2} - \frac{2\sigma^2}{\gamma+1}\right) + g_2\left(\frac{1}{2} + \frac{2\sigma^2}{\gamma+1}\right) + \frac{(1-g_2)}{4\sigma^2}J \quad (93)$$

$$G_{23} = \frac{(1-g_2)J}{\sigma^3} = \frac{F_2 J}{N\gamma} \quad (94)$$

$$R = 1 - \frac{3}{4}\left(\frac{\sqrt{z^2-1}}{\sqrt{\sigma^2-1}}\right) + \frac{2z^2+1}{4z}\frac{\ln\left(\frac{\bar{z}}{\bar{\sigma}}\right)}{\sqrt{\sigma^2-1}} \quad (95)$$

In which:

$\bar{\sigma} = \sigma + \sqrt{\sigma^2-1}$, $\bar{z} = z + \sqrt{z^2-1}$, $z = \theta/\delta$

and:

$$N = \frac{2\sigma^2}{(\sigma^2-1)(2\sigma^2+\gamma-1)} \quad (96)$$

$$J = \frac{2\sigma^2\left[\sigma^2 - 1 + (\gamma-1)\ln\sigma\right]}{(\sigma^2-1)(2\sigma^2+\gamma-1)} \quad (97)$$

$$F_1 = \frac{\gamma(1-g_1)N}{\sigma^3} \quad (98)$$

5. Calculating lift and drag forces

For a finite length of cone, the pressure force as shown in Fig. (4), is given by:

$$\vec{F} = -\iint_s p(\theta_C)\hat{n}_c \, ds \tag{99}$$

$$ds = R\,dr\,d\phi = z\tan\delta \frac{dz}{\cos\delta} d\phi \tag{100}$$

where, R, equals to the radius of a circle.

For pressure coefficient the perturbation expansion is defined by:

$$C_p = C_{p0} + \varepsilon C_{pm} \left(\frac{r}{l}\right)^m \cos n\phi + \alpha \widetilde{C}_{p1} \cos\phi + o(\varepsilon^2, \alpha^2, \alpha\varepsilon) \tag{101}$$

The expressions for lift and drag forces are:

$$dN = -1/2\,\rho_\infty V_\infty^2 C_p \tan\delta \cos\phi \, dz\, d\phi \tag{102}$$

$$dD = -1/2\,\rho_\infty V_\infty^2 C_p \tan^2\delta \, z dz\, d\phi \tag{103}$$

$$\frac{L}{D} = \frac{\alpha \widetilde{C}_{p1}}{2C_{p0}\tan\delta} \tag{104}$$

For comparing the lift to drag ratio in different cross sections first the relation between δ and the shape of cross section should be found. In rectangle Cartesian coordinates, an elliptic cone is represented by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{l} \tag{105}$$

where Cartesian to spherical transforms are:

$$x = r\sin\theta\cos\phi \tag{106}$$

$$y = r\sin\theta\sin\phi \tag{107}$$

$$z = r\cos\theta \tag{108}$$

Substituting Eqs. (106) to (108) into Eq. (105) the following relation is obtained:

$$\tan\theta = \frac{\tan\theta_m}{\sqrt{1 + e\cos 2\phi}} \tag{109}$$

where;

$$\tan\theta_m = \frac{\sqrt{2}ab}{\sqrt{a^2 + b^2}} = b\sqrt{1-e} \tag{110}$$

$$e = \frac{b^2 - a^2}{b^2 + a^2} \tag{111}$$

In the left hand side of Eq. (105) the Taylor expansion about $\varepsilon = 0$ is written and in right hand side the Fourier series are substituted. For different values of e calculations shows that Fourier series coefficients except for a_0 and a_2 are negligible, so the following equation is achieved:

$$\tan\delta - \varepsilon(1 + \tan^2\delta)\cos 2\phi = \tan\theta_m \left(\frac{a_0}{2} + a_2\cos 2\phi\right) \tag{112}$$

where;

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{1 + e\cos 2\phi}} d\phi \tag{113}$$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos 2\phi}{\sqrt{1 + e\cos 2\phi}} d\phi \tag{114}$$

It is obvious that for a circular cone, e=0 and:

$$\delta = b \text{ and } \varepsilon = 0 \tag{115}$$

Also elliptical cone with longitudinal curvature can be written as:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{l} \tag{116}$$

In spherical coordinate it is written as:

$$r\tan\theta\sin\theta = \frac{\tan^2\theta_m}{l(1 + e\cos 2\phi)} \tag{117}$$

And using Taylor expansion and Fourier series, the following relation is obtained:

$$r\left(\tan\delta\sin\delta - \varepsilon\left(\frac{r}{l}\right)^m \cos 2\phi \sin\delta(2 + \tan^2\delta)\right) = \frac{\tan^2\theta_m}{l(1 + e\cos^2\phi)} = \left(\frac{a_0}{2} + a_2\cos 2\phi\right)\tan^2\theta_m \tag{118}$$

Comparing the two sides of Eq. (118) the following expressions are achieved:

$$\frac{a_o}{2} = r\sin\delta\tan\delta \tag{119}$$

$$\tan\delta = \frac{-a_2 \pm \sqrt{a_2^2 - 8\varepsilon(\frac{r}{l})^m a_o/2}}{\varepsilon(\frac{r}{l})^m a_o} \tag{120}$$

In rectangle Cartesian coordinates, a cone with squircle cross section is represented by:

$$\frac{x^4}{z^4} + \frac{y^4}{z^4} = R^4 \tag{121}$$

Substitution of the Cartesian to spherical transforms in Eq. (121) gives:

$$\tan\theta_c = \frac{\sqrt{2}R}{(3+\cos 4\phi)^{1/4}} \tag{122}$$

Using Taylor expansion about $\varepsilon = 0$ for the left hand side of Eq. (122) and writing Fourier series for the right hand side, the following relation is obtained:

$$\tan\delta - \varepsilon(1+\tan^2\delta)\cos 4\phi = \frac{a_o}{2} + a_4\cos 4\phi \tag{123}$$

In comparison with a_0, a_4 the other coefficients of the Fourier series are much smaller and hence negligible. Comparing the two sides of Eq. (123) the following relations are achieved:

$$\tan\delta = \frac{a_o}{2} \tag{124}$$

$$\varepsilon = -\frac{a_4}{1+(\frac{a_o}{2})^2} \tag{125}$$

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sqrt{2}R}{(3+\cos 4\phi)^{1/4}} d\phi \tag{126}$$

$$a_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos 4\phi \frac{\sqrt{2}R}{(3+\cos 4\phi)^{1/4}} d\phi \tag{127}$$

In rectangle Cartesian coordinates, a cone with squircle cross section and longitudinal curvature can be represented by:

$$\frac{x^4}{R^4} + \frac{y^4}{R^4} = \frac{z}{l} \tag{128}$$

Substitution of the Cartesian to spherical transforms in Eq. (128) gives:

$$\frac{\sin^4\theta_C}{\cos\theta_C} = \frac{\sin^4\delta(1-\varepsilon(\frac{r}{l})^m\cos 4\phi)}{\cos\delta(1-\varepsilon(\frac{r}{l})^m\cos 4\phi)} \tag{129}$$

Using Taylor expansion about ε=0 for the left hand side of Eq. (129) and writing Fourier series for the right hand side, the following relations is obtained:

$$\tan\delta = \frac{-a_4 \pm \sqrt{a_4^2 - 16A^2}}{2A}, \quad A = \varepsilon(\frac{r}{l})^m \frac{a_0}{2} \tag{130}$$

$$a_0 = \frac{4}{\pi}\int_{-\pi}^{\pi} \frac{R^4}{l(3+\cos 4\phi)} d\phi \tag{131}$$

$$a_4 = \frac{4}{\pi}\int_{-\pi}^{\pi} \frac{R^4\cos 4\phi}{l(3+\cos 4\phi)} d\phi \tag{132}$$

6. Results and discussion

This analysis accounts the effects of cross section and longitudinal curvature by means of parameters that were named by *n* and *m* respectively. The aim of the present work is to improve lift to drag ratio by changing the cross section of the conical body. Using Fourier series a relation between δ and shape of the cross section of the body is obtained for each case. Figs. (5) and (6) represent the zero and first order of pressure coefficients as a function of k_δ ($k_\delta = M_\infty\delta$). Figures show these coefficients are constant in hypersonic limit. Fig. (7) shows the lift to drag ratio for different cross sections as a function of k_δ in α=4 (α is attack angle). A careful examination of Fig. (7) it reveals that the longitudinal curvature effect increases lift to drag ratio, also changing cross section from a circle to an ellipse and then to a squircle increases this ratio. For $k_\delta \to 0$, this ratio tends to zero, also as $k_\delta \to \infty$, a hypersonic limit is achieved. Increasing in attack angle can be seen in Fig. (8). The figure presents with increasing of attack angle the lift to drag ratio increases. Fig. (9) shows lift to drag ratio for different longitudinal curvature. It can be shown with increasing of m the longitudinal curvature (as shown in Fig. (3)) and then this ratio increases.

7. References

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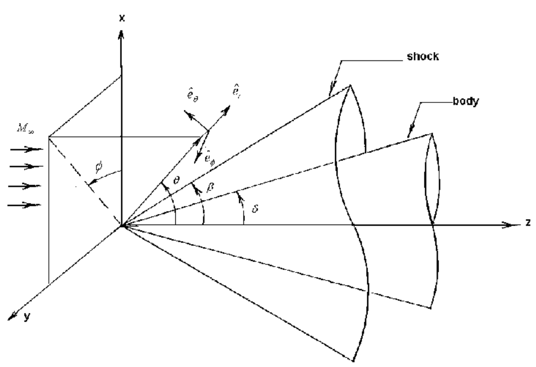

Fig. (1): The spherical coordinate of a cone.

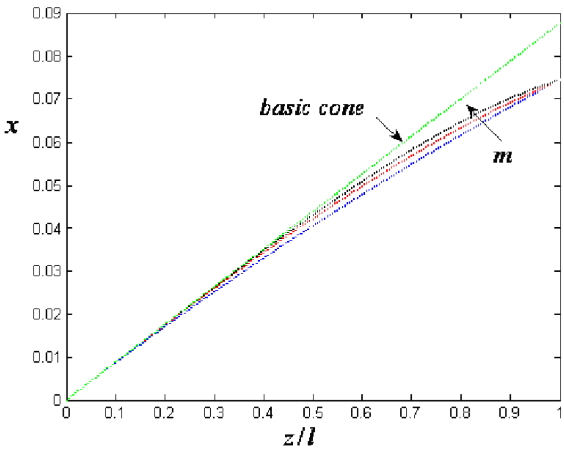

Fig. (2): Longitudinal curvature in m=1, 2,.

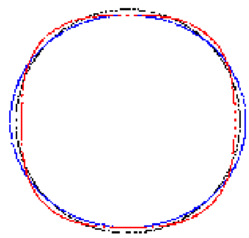

Fig. (3): Circular, elliptical and squircle cross section.

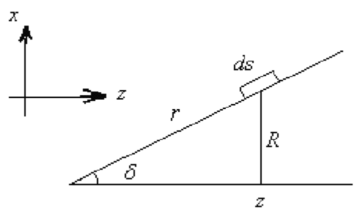

Fig.(4): Schematic of circular cone.

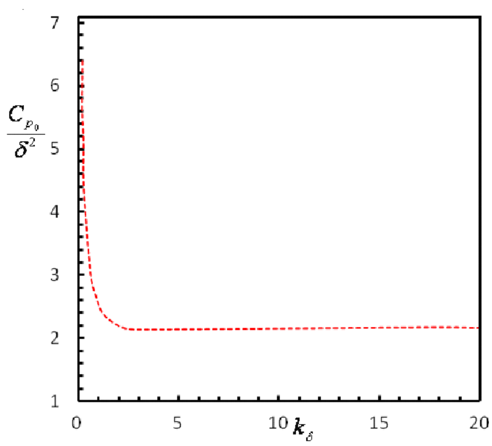

Fig. (5) Cp_o/δ^2 versus k_δ.

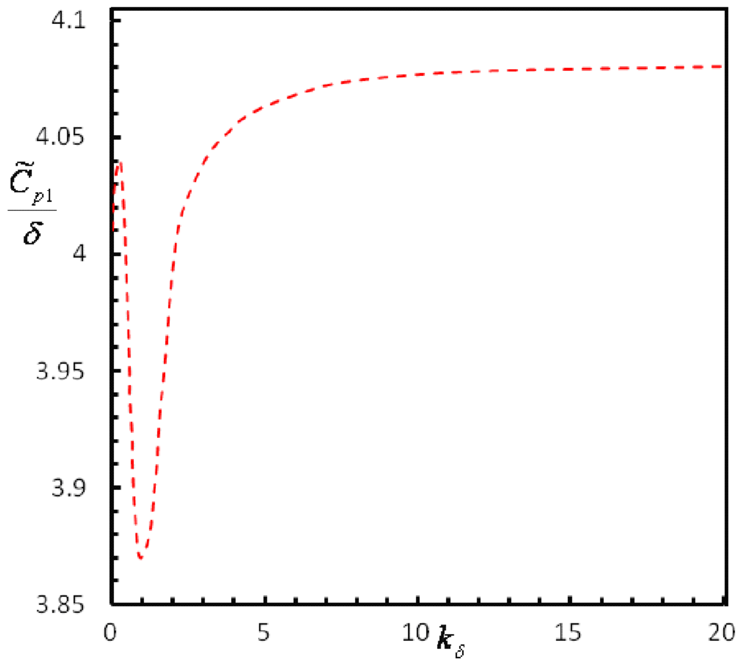

Fig. (6): Cp_1/δ versus k_δ.

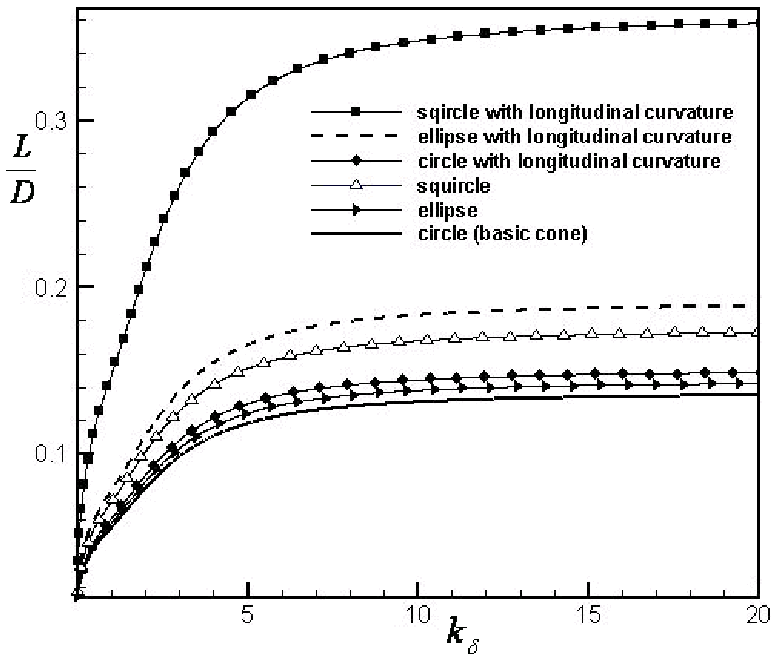

Fig.(7): Lift to drag ratio of k_δ for different cross sections in α=4.

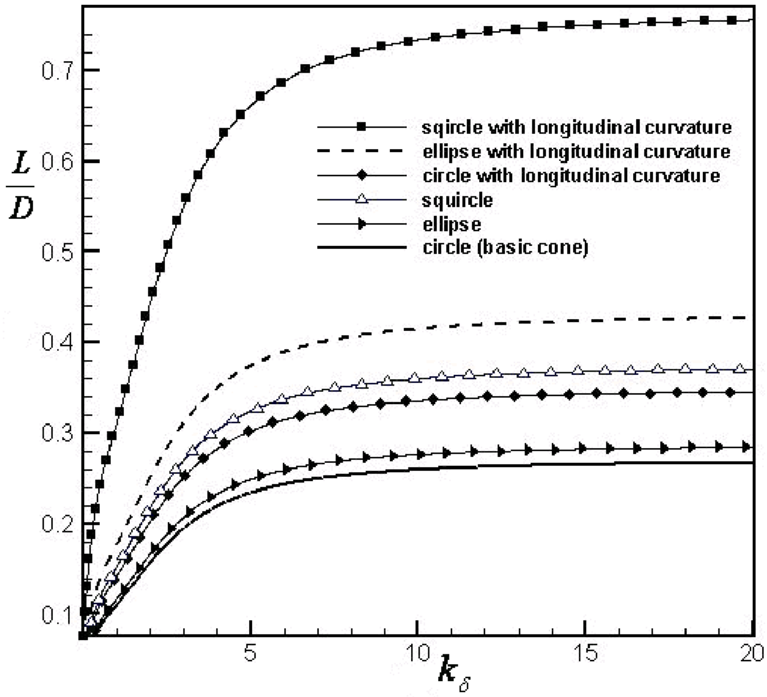

Fig.(8): lift to drag ratio of k_δ for different cross sections in α=8.

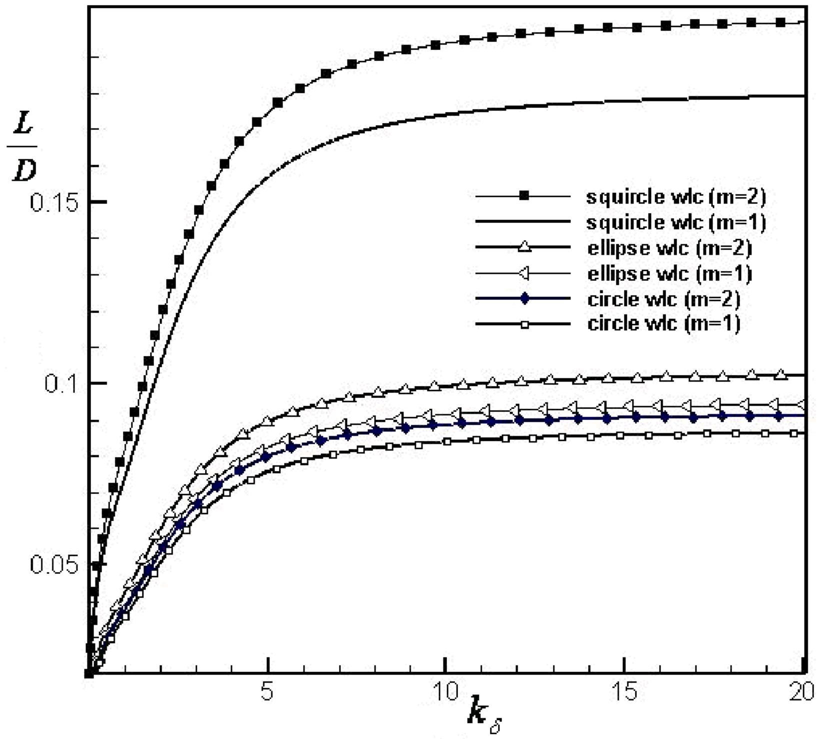

Fig. (9): Lift to drag ratio of k_δ for m =1, 2 (wlc: with longitudinal curvature).